

REMARKS ON SCALAR CURVATURE OF YAMABE SOLITONS

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ABSTRACT. In this paper, we consider the scalar curvature of Yamabe solitons. In particular we show that, with natural conditions and non positive Ricci curvature, any complete Yamabe soliton has constant scalar curvature, namely, it is a Yamabe metric. We also show that the quadratic decay at infinity of the Ricci curvature of a complete non-compact Yamabe soliton has non-negative scalar curvature. A new proof of Kazdan-Warner condition is also presented.

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1. INTRODUCTION

In this work, we study the special solutions, the so called the Yamabe solitons, to the Yamabe flow, which was introduced by R.Hamilton at the same time as Ricci flow. We note that the Yamabe flow has some similar properties as Ricci flow ([7],[11], [10][6],[4]). Since the Yamabe solitons come naturally from the blow-up procedure along the Yamabe flow [1][4][3][7], we are lead to study the Yamabe solitons on complete non-compact Riemannian manifolds. We shall study some properties of the scalar curvature of the Yamabe solitons on complete non-compact Riemannian manifolds. Recall that a Riemannian manifold (M, g) is called a Yamabe soliton if there are a smooth vector field X and constant ρ such that

$$(1) \quad (R - \rho)g = \frac{1}{2}L_X g \quad \text{on } M,$$

where R is the scalar curvature and $L_X g$ is the Lie derivative of the metric g ; When $X = \nabla f$ for some smooth function f , we call it the gradient Yamabe soliton. The function f above will be called the potential function and it is determined up to a constant. In this case the equation (1) becomes

$$(2) \quad (R - \rho)g = \nabla^2 f \quad \text{on } M,$$

When the constant $\rho \geq 0$, we call the Yamabe solitons the non-expanding Yamabe solitons.

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In this paper we shall prove the following results on the sign of the scalar curvature R of a Yamabe soliton depending on some asymptotic behaviour of it.

Theorem 1. *Let (M, g) be a complete and non-compact gradient Yamabe soliton with $\rho \geq 0$. Assume that $\underline{\lim}_{x \rightarrow \infty} R(x) \geq 0$. Then the scalar curvature R of (M, g) is non-negative. Furthermore, if (M, g) is not scalar flat, then $R > 0$ on M .*

We shall use the argument from [9] to get another results about non-expanding Yamabe solitons.

Theorem 2. *Let (M, g) be a complete and non-compact gradient Yamabe soliton with $\rho \geq 0$. Assume that there is some point x_0 such that for some large uniform constant $R_0 > 1$,*

$$\int_{\gamma} [R - 2(n-1)\text{Ric}(\gamma', \gamma')] \leq \rho d(x),$$

for any minimizing geodesic curve γ connecting x_0 to x with $d(x, x_0) \geq R_0$. Then $R \geq 0$.

The proof of this theorem will be given in section 4.

We can show that in some cases the Yamabe solitons are Yamabe metrics.

Theorem 3. *Let (M, g) be a complete and non-compact gradient Yamabe soliton such that $|R - \rho| \in L^1(M)$, $\int_M \text{Ric}(\nabla f, \nabla f) \leq 0$ and the potential function f has at most quadratic growth on M ; that is,*

$$|f(x)| \leq Cd(x, x_0)^2, \quad |\nabla f| \leq C(1 + d(x, x_0)^2),$$

near infinity, where C is some uniform constant and $d(x, x_0)$ is the distance function from the point x to a fixed point x_0 . Then $R = \rho$ on (M, g) .

We shall also study Liouville type theorem of harmonic functions with finite Dirichlet integral. We now show the following result.

Theorem 4. *Let (M, g) be a complete and non-compact Riemannian manifold with non-negative Ricci curvature. Assume that u is a harmonic function with finite weighted Dirichlet integral, i.e., for some ball $B(x_0)$,*

$$\int_{M-B(x_0)} d(x, x_0)^{-2} |\nabla u|^2 < \infty.$$

Then $\nabla^2 u = 0$ on M .

Then, use the idea of the proof of the above result to study the Yamabe solitons and we obtain

Theorem 5. *Assume that the Yamabe soliton (M, g, X) has non-positive Ricci curvature. Suppose that*

$$(3) \quad \int_{M-B(x_0)} d(x, x_0)^{-2} |X|^2 < \infty.$$

Then

$$\nabla X = 0$$

and $R = \rho$.

Let us remark that Theorem 5 applies to solitons in general, they do not need to be gradient nor non-expanding. When applied to non-expanding solitons, Theorem 5 states that the only non-expanding solitons with non positive Ricci curvature and satisfying condition (3) are the Ricci-flat and steady ones.

2. PROOFS OF THEOREMS 1, 4 AND 5

Proof. of Theorem 1.

We shall denote $Ric = (R_{ij})$ the Ricci tensor in the local coordinates (x^j) .

First we shall obtain a formula for the laplacian os the scalar curvature of a gradient Yamabe soliton. Taking the k-derivative to (2) we have

$$\nabla_k f_{ij} = \nabla_k R g_{ij}.$$

Using the Ricci formula [2] we get that

$$\nabla_i f_{jk} + R_{jikl} f_l = \nabla_k R g_{ij}.$$

By contraction for j, k ,

$$\nabla_i \Delta f + R_{il} f_l = \nabla_i R.$$

Then we have

$$n R_i + R_{il} f_l = R_i.$$

Then we have

$$-R_{il} f_l = (n - 1) R_l,$$

or write in another way

$$(4) \quad -Ric(\nabla f, \cdot) = (n - 1) \nabla R.$$

Taking one more derivative we have

$$(n - 1) \Delta R = -R_{il,i} f_l - R_{il} f_{il}.$$

Recall the contracted Bianchi identity that

$$R_{il,i} = \frac{1}{2} R_l.$$

Then we have

$$(n - 1) \Delta R = -\frac{1}{2} (\nabla R, \nabla f) - R(R - \rho).$$

we have

$$(5) \quad (n - 1) \Delta R + \frac{1}{2} g(\nabla f, \nabla R) + R^2 - \rho R = 0.$$

Using the maximum principle we can prove Theorem 1. Assume that $\inf_M R(x) < 0$. Since $\lim_{x \rightarrow \infty} R(x) \geq 0$, we know that there is some point $z \in M$ such that $R(z) = \inf_M R(x) < 0$. Then we have

$$\Delta R(z) \geq 0, \quad \nabla R(z) = 0.$$

By this we have at z that

$$(n-1)\Delta R + \frac{1}{2}g(\nabla f, \nabla R) \geq 0$$

and by (5),

$$R(z)^2 - \rho R(z) \leq 0.$$

This is absurd since $R(z)^2 - \rho R(z) > 0$ for $\rho \geq 0$. The strong maximum principle implies that either $R(x) > 0$ or $R(x) = 0$ on M . \square

Proof. of Theorem 4

Recall the Bochner formula [2] that

$$(6) \quad \frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + g(\nabla u, \nabla \Delta u) + Ric(\nabla u, \nabla u).$$

Then using the harmonicity of u , we have

$$|\nabla^2 u|^2 + Ric(\nabla u, \nabla u) = \frac{1}{2}\Delta|\nabla u|^2.$$

Choose a cut-off function $\phi = \phi_r$ on the ball $B_{2r}(x_0)$, where $r > 0$ (and we let $B_r = B_r(x_0)$ for simplicity) such that

$$\phi_r = 1, \quad \text{in } B_r; \quad |\nabla \phi_r|^2 \leq \frac{C}{r^2},$$

and

$$\Delta \phi_r \leq \frac{C}{r^2}.$$

These imply that

$$\Delta \phi_r^2 \leq \frac{C}{r^2} \rightarrow 0$$

as $r \rightarrow \infty$. Then we have

$$\int [|\nabla^2 u|^2 + Ric(\nabla u, \nabla u)] \phi_r^2 = \int \frac{1}{2}\Delta|\nabla u|^2 \phi_r^2.$$

Using the integration by part and our assumption, we have

$$\int \frac{1}{2}\Delta|\nabla u|^2 \phi_r^2 = \int \frac{1}{2}|\nabla u|^2 \Delta \phi_r^2,$$

which is, by our assumption,

$$\leq \int_{B_{2r}-B_r} \frac{C}{2r^2} |\nabla u|^2 \rightarrow 0.$$

as $r \rightarrow \infty$. Hence we have

$$\int_M [|\nabla^2 u|^2 + Ric(\nabla u, \nabla u)] = 0,$$

which implies that $\nabla^2 u = 0$ and $Ric(\nabla u, \nabla u) = 0$ on M . \square

We now use the idea above to study the Yamabe solitons and give the *Proof.* of Theorem 5. By taking the trace, from the defining equation of Yamabe soliton, we have that a Yamabe soliton satisfies

$$(7) \quad \operatorname{div} X = n(R - \rho), \quad \text{on } M.$$

Recall the following Bochner formula [12]

$$\operatorname{div}(L_X g)(X) = \frac{1}{2} \Delta |X|^2 - |\nabla X|^2 + \operatorname{Ric}(X, X) + \nabla_X \operatorname{div}(X).$$

Then we have

$$(8) \quad |\nabla X|^2 = \frac{1}{2} \Delta |X|^2 + \operatorname{Ric}(X, X) + (n - 2) \nabla_X R.$$

Fixing a cut-off function ϕ as above, we then have that

$$\int X_j \nabla_j R \phi^2 = - \int \operatorname{div} X (R - \rho) \phi^2 + 2 \phi \nabla_X \phi (R - \rho).$$

Hence

$$\int \nabla_X R \phi^2 = -n \int (R - \rho)^2 \phi^2 - 2 \int \phi \nabla_X \phi (R - \rho).$$

Integrating (8) we have

$$\int |\nabla X|^2 \phi^2 = \frac{1}{2} \int (\Delta \phi^2) |X|^2 + \int \operatorname{Ric}(X, X) \phi^2 + (n - 2) \int \nabla_X R \phi^2.$$

Then we obtain

$$\begin{aligned} \int |\nabla X|^2 \phi^2 + n(n - 2) \int (R - \rho)^2 \phi^2 &= \frac{1}{2} \int (\Delta \phi^2) |X|^2 \\ &+ \int \operatorname{Ric}(X, X) \phi^2 - 2(n - 2) \int \phi \nabla_X \phi (R - \rho). \end{aligned}$$

Using the Young and Cauchy-Schwartz inequalities we can get that

$$\begin{aligned} \int |\nabla X|^2 \phi^2 + (n - 1)(n - 2) \int (R - \rho)^2 \phi^2 \\ \leq \frac{1}{2} \int (\Delta \phi^2) |X|^2 + \int \operatorname{Ric}(X, X) \phi^2 + C(n) \int |X|^2 |\nabla \phi|^2 \end{aligned}$$

for some uniform constant $C(n)$.

Then we have proved theorem 5. \square

3. PROOFS OF THEOREM 3 AND RELATED RESULTS

The argument of Theorem 3 follows from the following proposition (see also [5]).

Proposition 6. *Let (M, g) be a Yamabe soliton with smooth boundary. Then we have*

$$n(n - 1) \int_M (R - \rho)^2 - \int_M \operatorname{Ric}(\nabla f, \nabla f) = (n - 1) \int_{\partial M} (R - \rho) \nabla_\nu f.$$

where ν is the outward unit normal to the boundary ∂M .

Proof. We use the argument from [8] (see also [5]). Note that

$$\int_M |\Delta f|^2 = \int \Delta f f_{jj}.$$

Integrating by parts we get that

$$\int \Delta f f_{jj} = \int_{\partial M} \Delta f \nabla_\nu f - \int \nabla \Delta f \cdot \nabla f.$$

Then using the Bochner formula (6) we have

$$\int \Delta f f_{jj} = \int_{\partial M} \textcolor{red}{n} (R - \rho) \nabla_\nu f + \int_M \left(|\nabla^2 f|^2 + Ric(\nabla f, \nabla f) - \frac{1}{2} \Delta |\nabla f|^2 \right).$$

Note that

$$\begin{aligned} \int_M \Delta |\nabla f|^2 &= \int_{\partial M} \nabla_\nu |\nabla f|^2 = \int_{\partial M} 2 \langle \nabla_\nu \nabla f, \nabla f \rangle \\ &= 2 \int_{\partial M} \nabla^2 f \langle \nu, \nabla f \rangle = 2 \int_{\partial M} (R - \rho) \langle \nu, \nabla f \rangle. \end{aligned}$$

Then we have

$$\int_M |\Delta f|^2 = \int_M (|\nabla^2 f|^2 + Ric(\nabla f, \nabla f)) + (n-1) \int_{\partial M} (R - \rho) \nabla_\nu f.$$

And, using (2) in the above formula, we obtain

$$(n^2 - n) \int_M (R - \rho)^2 - \int_M Ric(\nabla f, \nabla f) = (n-1) \int_{\partial M} (R - \rho) \nabla_\nu f.$$

□

We now *prove Theorem 3*.

Proof. By proposition 6, we know that for the dimension constant $C_n > 0$,

$$C_n \int_{B_r} |R - \rho|^2 - \int_M Ric(\nabla f, \nabla f) = (n-1) \int_{\partial B_r} (R - \rho) \nabla_\nu f \leq Cr \int_{\partial B_r} |R - \rho|.$$

We now choose $r = r_j \rightarrow \infty$ such that

$$r \int_{\partial B_r} |R - \rho| \rightarrow 0.$$

This is obtained by using the fact that $\int_M |R - \rho| < \infty$ and Fubini's theorem. Then, when $\int_M Ric(\nabla f, \nabla f) \leq 0$, we have

$$\int_M |R - \rho|^2 = 0,$$

and this implies that $R = \rho$ on M . □

We take this chance to give another proof of Kazdan-Warner condition; namely,

Proposition 7. *Assume that X is a conformal vector field on the compact Riemannian manifold (M, g) , i.e., there exists a smooth function $a(x)$ on M such that*

$$L_X g = a(x)g.$$

Then we have

$$\int_M \nabla_X R dv_g = -\frac{2n}{n-2} \int_{\partial M} (Ric - \frac{R}{n}g)(\nu, X) d\sigma_g,$$

where ν is the outer unit normal to the boundary ∂M .

Proof. Set

$$\dot{Ric} = Ric - \frac{R}{n}g.$$

Then by the contracted Bianchi identity we get

$$\delta \dot{Ric} = -\frac{n-2}{2n} dR.$$

We now compute

$$\int_M \nabla_X R dv_g = -\frac{2n}{n-2} \int_M \delta \dot{Ric}(X) dv_g.$$

Integrating by part we get that

$$\int_M \delta \dot{Ric}(X) dv_g = - \int_M (Ric, \nabla X) dv_g + \int_{\partial M} Ric(\nu, X) d\sigma_g.$$

We then have

$$\int_M \delta \dot{Ric}(X) dv_g = \int_{\partial M} Ric(\nu, X) d\sigma_g - \frac{1}{2} \int_M (Ric, L_X g) dv_g.$$

Recall that

$$\frac{1}{2} L_X g = \frac{1}{2} a(x)g.$$

Since $(Ric, g) = 0$, we obtain that

$$\int_M \delta \dot{Ric}(X) dv_g = \int_{\partial M} Ric(\nu, X) d\sigma_g.$$

This completes the proof of proposition 7.

4. PROOF OF THEOREM 2

The proof of Theorem 2 will follow the argument of pseudo-locality theorem due to Perelman [9]. The idea of proof of Theorem 2 is similar to Perelman's Li-Yau harnack differential inequality. To make it, we recall some well-known facts.

Define $d(x) = d(x, x_0)$. Let $\gamma(s)$ ($\gamma : [0, (x)] \rightarrow M$) be a shortest geodesic curve from x_0 to x . Without loss of generality, we may assume that the distance function $d(x)$ is smooth at x . Choose an orthonormal basis (e_1, e_2, \dots, e_n) at x_0 with $e_1 = \gamma'(0)$. Extend the basis into a parallel basis $(e_1(\gamma(s)), e_2(\gamma(s)), \dots, e_n(\gamma(s)))$ along the curve $\gamma(s)$. Let $X_j(s)$ be the

Jacobian vector field along $\gamma(s)$ with $X_j(0) = 0$ and $X_j(d(x)) = e_j(d(x))$. Then we have

$$\Delta d(x) = \sum_j \int_0^{d(x)} (|X'_j(s)|^2 - R(\gamma', X_j, \gamma', X_j)) ds.$$

Fix some $r_0 > 0$ such that $|Ric| \leq (n-1)K$ on $B_{r_0}(x_0)$. Define

$$Y_j(s) = a_j(s)e_j(s)$$

for $j \geq 2$, where $a_j(s)$ is $\frac{s}{r_0}$ on $[0, r_0]$ and $a_j(s) = 1$ on $[r_0, d(x)]$.

Using the minimizing property of the Jacobi field we have

$$\begin{aligned} & \sum_j \int_0^{d(x)} (|X'_j(s)|^2 - R(\gamma', X_j, \gamma', X_j)) ds \\ & \leq \sum_j \int_0^{d(x)} (|Y'_j(s)|^2 - R(\gamma', Y_j, \gamma', Y_j)) ds. \end{aligned}$$

By direct computation [9] we have

$$\begin{aligned} & \sum_j \int_0^{d(x)} (|Y'_j(s)|^2 - R(\gamma', Y_j, \gamma', Y_j)) ds \\ & = - \int_0^{d(x)} Ric(\gamma', \gamma') + \int_0^{r_0} \left(\frac{n-1}{r_0^2} + \left(1 - \frac{s^2}{r_0^2}\right) Ric(\gamma', \gamma') \right) ds \end{aligned}$$

and the latter is less than

$$- \int_\gamma Ric(\gamma', \gamma') + (n-1) \left(\frac{2}{3} K r_0 + \frac{1}{r_0} \right).$$

It is easy to see that

$$g(\nabla f, \nabla d) = \nabla_{\gamma'} f(x) \leq \int_\gamma \nabla^2 f(\gamma', \gamma') + |\nabla f(x_0)|.$$

Using

$$\nabla^2 f(\gamma', \gamma') = R - \rho$$

we then have

$$g(\nabla f, \nabla d) \leq -\rho d(x) + \int_\gamma R + |\nabla f(x_0)|.$$

Hence, we have, for some uniform constant $C > 0$,

(9)

$$2(n-1)\Delta d(x) + g(\nabla f, \nabla d) \leq -\rho d(x) + \int_\gamma [-2(n-1)Ric(\gamma', \gamma') + R] + C/r_0^2.$$

We may choose r_0 such that the latter is less than $\frac{4(n-1)}{r_0^2}$.

For any fixed $A > 2$ we shall consider the new function

$$u(x) = \phi\left(\frac{d(x)}{A r_0}\right) R(x),$$

where ϕ is the cut-off function on the real line \mathbb{R} defined after formula (6), with $r = Ar_0$. We denote by $D = \frac{\phi''}{\phi}$ and $h = \frac{\phi'}{\phi}$.

We compute

$$\Delta u(x) = R\Delta\phi + 2g(\nabla R, \nabla\phi) + \phi\Delta R.$$

Note that $u = 0$ outside the ball of radius $2Ar_0$.

It is clear that if $\inf_M u = 0$ for every A , then we have $R \geq 0$ on M .

If $\inf_M u < 0$ for some $A = A_0$, then $\inf_M u < 0$ for every $A > A_0$, and there is some point $x_1 \in B_{2Ar_0}(x_0)$ such that

$$u(x_1) = \inf_M u < 0.$$

Then we have $R(x_1) < 0$. By this we have

$$\phi'(x_1)R(x_1) > 0,$$

which implies $x_1 \notin B_{Ar_0}(x_0)$. Moreover, at the minimum x_1 ,

$$(10) \quad \nabla u = 0, \quad \Delta u \geq 0$$

The following differential inequality by now is more or less a standard computation (see [9]), but we shall give the details for the convenience of the reader. Using these two properties (10) and the equations (5) and (9) we can get that

$$\begin{aligned} \Delta u(x_1) &= \left(\frac{D}{(Ar_0)^2} + \frac{h}{Ar_0} \Delta d \right) u(x_1) + \frac{1}{2(n-1)} \frac{h}{Ar_0} (\nabla f, \nabla d) u(x_1) \\ &+ \frac{1}{n-1} \rho u(x_1) - \phi R^2 - 2h^2 \frac{1}{(Ar_0)^2} u(x_1) \\ &\leq \left(\frac{D}{(Ar_0)^2} - \frac{2h^2}{(Ar_0)^2} \right) u(x_1) - \frac{1}{n-1} \phi R^2 \\ &+ \frac{h}{Ar_0} [\Delta d + \frac{1}{2(n-1)} \langle \nabla f, \nabla d \rangle] u(x_1) \\ &\leq \left(\frac{D}{(Ar_0)^2} - \frac{2h^2}{(Ar_0)^2} \right) u(x_1) - \frac{1}{(n-1)\phi} u(x_1)^2 + \frac{2h}{(Ar_0)^2} u(x_1). \end{aligned}$$

Then

$$\Delta u(x_1) \leq \frac{|u(x_1)|}{\phi} \left\{ \frac{1}{A^2 r_0^2} \left[\frac{2\phi'^2}{\phi} + 2|\phi'| + |\phi''| \right] - \frac{1}{n-1} |u(x_1)| \right\}.$$

For some uniform constant $C > 0$, we have

$$2|\phi'| \leq C, \quad \frac{2\phi'^2}{\phi} \leq C, \quad |\phi''| \leq C,$$

Then we can show that

$$|u(x_1)| \leq \frac{(n-1)C}{A^2 r_0^2}.$$

The latter implies that

$$R(x) \geq -\frac{(n-1)C}{A^2 r_0^2} \quad \text{on } B_{2Ar_0}(x_0).$$

Sending $A \rightarrow \infty$, we get that $R \geq 0$ on M .

This completes the proof of Theorem 2.

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